Int J Theor Phys (2009) 48: 2432–2440
DOI 10.1007/s10773-009-0033-y

Feynman's Proof of Maxwell Equations: in the Context of Quantum Gravity

P. Narayana Swamy

Received: 24 March 2009 / Accepted: 6 May 2009 / Published online: 20 May 2009

Abstract Many of us are familiar with Feynman's "proof" of 1948, as revealed by Dyson, which demonstrates that Maxwell equations of electromagnetism are a consequence of Newton's laws of motion of classical mechanics and the commutation relations of coordinate and momentum of quantum mechanics. It was Feynman's purpose to explore the universality of dynamics of particles while making the fewest assumptions. We re-examine this formulation in the context of quantum gravity and show how Feynman's derivation can be extended to include quantum gravity.

1 Introduction

Feynman's proof in 1948 of the Maxwell equations has been put on record by Dyson in 1990 [1]. This work by Feynman is a derivation of Maxwell equations assuming only Newton's laws of motion and the quantum Heisenberg relation between position and velocity of a single non-relativistic particle. We shall re-examine this derivation, extending the proof to the case when the Heisenberg uncertainty relation in the form of the fundamental commutation relation is generalized in the context of quantum gravity. Rather than the original form of the proof employing the variables $x_i, \dot{x}_i, \ddot{x}_i$, and Newton's second law of motion, we shall find it more expedient to follow the Lagrangian method of classical mechanics for this purpose, as described in Sect. 2 of Dyson's paper. We shall first summarize this derivation below.

We begin with the commutation relations of quantum mechanics

$$[x_j, x_k] = 0, \qquad m[x_j, \dot{x}_k] = i\hbar\delta_{jk}, \qquad [x_j, P_k] = i\hbar\delta_{jk}. \tag{1}$$

If one defines the vector potential by

$$P_k = m\dot{x}_k + A_k, \tag{2}$$

where $\dot{x}_k$ is the velocity of the particle, it then follows that the vector potential satisfies $[x_j, A_k] = 0$ which implies that it is independent of velocity and depends only on x, t. Then

P.N. Swamy (✉)
Department of Physics, Southern Illinois University, Edwardsville, IL 62026, USA
e-mail: pswamy@siue.edu

one considers the Lagrange equations

$$P_k = \frac{\partial \mathcal{L}}{\partial \dot{x}_k}, \qquad \dot{P}_k = \frac{\partial \mathcal{L}}{\partial x_k}, \tag{3}$$

where the Lagrangian is $\mathcal{L} = \mathcal{L}(x, \dot{x}, t)$. The argument uses the fact that

$$\dot{A}_k = \frac{dA_k}{dt} = \frac{\partial A_k}{\partial t} + \dot{x}_j \frac{\partial A_k}{\partial x_j}. \tag{4}$$

Integrating the first equation above determines the Lagrangian as

$$\mathcal{L} = \frac{1}{2} m \dot{x}_k \dot{x}_k + \dot{x}_k A_k + \phi \tag{5}$$

where ϕ is independent of velocity. The second equation determines the force as $m\ddot{x}$. Indeed one obtains

$$F_j = E_j + \epsilon_{jkl} \dot{x}_k B_l, \tag{6}$$

which defines the electric and magnetic fields as

$$\mathbf{B} = \nabla \times \mathbf{A}, \qquad \mathbf{E} = \nabla\phi - \frac{\partial \mathbf{A}}{\partial t}. \tag{7}$$

The Maxwell equations

$$\nabla \cdot \mathbf{B} = 0, \qquad \nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0. \tag{8}$$

follow immediately. As pointed out by Dyson [1], the other two Maxwell equations

$$\nabla \cdot \mathbf{E} = 4\pi\rho, \qquad \frac{\partial \mathbf{E}}{\partial t} - \nabla \times \mathbf{B} = 4\pi \mathbf{J}, \tag{9}$$

merely define the external charge and current densities and hence are not derived. Feynman wanted to deal only with the fields and their properties and not the sources. This is the end of Feynman's "proof".

It is important to stress that according to the editorial comment of Dyson [1], "the recently educated young physicists would say that the result is trivial, that the proof is unnecessarily complicated"; hence Dyson put the proof in the proper historical context.

It must be pointed out that Feynman's intent was to make as few assumptions as possible so that we do not get caught up in the intricacies of the theoretical formulation, with the purpose of exploring the universe of particle dynamics as widely as possible. Feynman's point of view was indeed different, as he was exploring possible alternatives to the standard formulation. He wanted to ensure that the starting assumptions were as unrestrictive as possible, in particular, he was not pursuing any model building. One must admit, as pointed out by Dyson [1], that there are inconsistencies. The Maxwell equations are relativistically covariant while the Newtonian assumptions used by Feynman are non-relativistic. Yet, according to Dyson, Feynman's derivation indeed raises questions but nevertheless worth studying. It must be further observed that the intent of the approach is certainly not to demonstrate that the Maxwell equations of classical electromagnetism are a consequence of the fundamentals of quantum mechanics. Hence it must be observed that Feynman's intent was rather to understand the connection, if any, between the universe of particle dynamics and the Heisenberg uncertainty relations of quantum physics.

Accordingly, it is worthwhile to extend Feynman's derivation to include quantum gravity i.e., include a fundamental length arising from gravity, in order to fully understand the connection of particle dynamics and quantum mechanics with Maxwell equations. Such a connection, if it exists, demonstrates a consistency of premises rather than an argument that one is a consequence of another. In other words, we address the question: is there a connection between the generalized Heisenberg uncertainty relations of quantum gravity and classical non-relativistic dynamics, in the sense of Feynman? We shall now examine such an extension. We shall show that such an extension is rather straightforward.

2 Original *Ansatze* in the Context of Quantum Gravity

We begin by defining the vector potential by the relation

$$P_k = m\dot{x}_k + A_k \tag{10}$$

where we have the effective momentum on the left hand side, the canonical momentum of the Lagrangian formulation. The basic premise consists of the following quantum commutation relations

$$[x_j, x_k] = 0, \qquad m[x_j, \dot{x}_k] = i\hbar\delta_{jk}(1 + \beta P_m P_m) \tag{11}$$

together with the quantum commutation relation

$$[x_j, P_k] = i\hbar\delta_{jk}(1 + \beta P_m P_m). \tag{12}$$

The parameter $\beta = L_P^2/\hbar^2 = G/\hbar c^3$, appearing in both the above equations is positive and independent of x_i and P_i. It implies a minimum length described by the parameter β, as will be described later. In particular, equation (12) implies the uncertainty relation

$$\Delta x_j \Delta P_k \geq \frac{1}{2}\hbar\delta_{jk}(1 + \beta(\Delta P)^2). \tag{13}$$

This generalized uncertainty relation has appeared in the context of quantum gravity and string theory and implies that a minimal length should quantum theoretically be described [2] [3, 4, see additional references contained therein] [5–9] as a minimal uncertainty in position measurements. The notion of a fundamental length and the connection to quantum gravity is well-known. For instance, Smith's text on quantum mechanics [10], the Introductory chapter contains an account of how to define the fundamental length, fundamental mass and fundamental time in terms of powers of the fundamental constants $\hbar$, G and the speed of light in vacuum c. The only length that can be obtained is defined by

$$L_P = \sqrt{\frac{\hbar G}{c^3}}, \tag{14}$$

and commonly identified as the Planck length. On the subject of detectors of gravitational radiation, Ohanian et al. [11] illustrate the appearance of $\delta x = \sqrt{\hbar\tau/m}$, in terms of τ, the characteristic time in quantum gravity, which can be expressed as $(L/c)^2(GE/c^2r^2)$ as explained below. This will accordingly add to the uncertainty in the Heisenberg uncertainty relation. A thorough discussion and explanation of the basic role played by gravitation in the cause of a fundamental length in nature is contained in the work of Mead [12, other references cited therein]. This work also contains a derivation of the uncertainty produced by quantum gravity. The work of Kempf et al. [5] illustrates the role of a fundamental length in the deformed uncertainty relation. The work of Adler et al. [7, 8] establishes the deformed uncertainty relation in several rigorous theories of quantum gravity. While it suffices to quote this reference to Adler's work rather than describe the details of this connection, it may indeed be worthwhile to provide a brief summary of this idea.

Let us begin with Heisenberg's uncertainty relation

$$\Delta x \Delta p \approx \hbar, \tag{15}$$

and introduce the gravitational interaction between the photon and the electron. The photon behaves as a classical particle with an effective mass equal to E/c^2, The electron is in an experimental region described by a typical size L, inside of which it interacts with the photon and thus will experience an acceleration due to gravity described by the equation of motion

$$\ddot{\mathbf{r}} = -\frac{GE}{c^2 r^2}\hat{r}, \tag{16}$$

where G is the constant of gravitation and r is the distance between the electron and the photon. The size L is taken to be a characteristic length describing the system. During this gravitational interaction, the electron will acquire a velocity and consequently a displacement described by the equations

$$\Delta v \approx \frac{GE}{c^2 r^2}\left(\frac{L}{c}\right), \qquad \Delta x \approx \frac{GE}{c^2 r^2}\left(\frac{L}{c}\right)^2. \tag{17}$$

Accordingly, the uncertainty in the position of the electron is

$$\Delta x \approx \frac{G\Delta p}{c^3}. \tag{18}$$

We may express this in terms of the Planck length L_P, a parameter customarily defined in quantum gravity by $L_P^2 = G\hbar/c^3$. Accordingly, the uncertainty in the position of the electron becomes

$$\Delta x \approx L_P^2 \frac{\Delta p}{\hbar}. \tag{19}$$

Hence the modified uncertainty relation is

$$\Delta x \approx \frac{\hbar}{\Delta p} + L_P^2 \frac{\Delta p}{\hbar}. \tag{20}$$

Thus the standard uncertainty relation is modified to include gravity in this manner and we might refer to this as the extended uncertainty principle.

For the sake of expediency, we shall regard the presence of the additional term containing the parameter β as due to a deformation as a result of quantum gravity and a fundamental length so that the case of $\beta \to 0$ would correspond to the undeformed case. We stress that the deformation in (11, 12) are identical. This is in accordance with Dyson's [1] formulation where both the commutation relations are identical and we shall investigate the consequences of this premise. It follows from (11, 12) that

$$[x_j, A_k] = 0, \tag{21}$$

and this immediately implies that A_k is independent of $\dot{x}_j$ and depends only on the coordinate and time. We shall now consider the Lagrange equation

$$P_k = \frac{\partial \mathcal{L}}{\partial \dot{x}_k}, \tag{22}$$

which, upon integration yields the Lagrangian

$$\mathcal{L} = \frac{1}{2} m \dot{x}_l \dot{x}_l + \dot{x}_k A_k + \phi, \tag{23}$$

where ϕ is a "constant" of integration independent of $\dot{x}$ and depends only on the coordinate and time. We shall next consider the Lagrange equation

$$\dot{P}_k = \frac{\partial \mathcal{L}}{\partial x_k}. \tag{24}$$

From this it follows that

$$\dot{P}_k = m\ddot{x}_k + \frac{dA_k}{dt} = \dot{x}_l \frac{\partial A_l}{\partial x_k} + \frac{\partial \phi}{\partial x_k}. \tag{25}$$

This enables us to define the force as

$$F_k = \frac{\partial \phi}{\partial x_k} - \frac{\partial A_k}{\partial t} - \frac{\partial A_k}{\partial x_l}\dot{x}_l + \dot{x}_l \frac{\partial A_l}{\partial x_k}. \tag{26}$$

We may then define the electric field and the magnetic field as

$$E_k = \frac{\partial \phi}{\partial x_k} - \frac{\partial A_k}{\partial t}; \qquad d\epsilon_{klm} B_m = \frac{\partial A_l}{\partial x_k} - \frac{\partial A_k}{\partial x_l}. \tag{27}$$

The above equation may be re-expressed as

$$B_m = \epsilon_{klm} \frac{\partial A_l}{\partial x_k}. \tag{28}$$

We observe that both the electric and magnetic fields are independent of $\dot{x}$ and depend only the coordinate and time as a result of the fact that A_k and ϕ are independent of the velocity. This immediately leads to the two Maxwell equations of electromagnetism

$$\nabla \cdot \mathbf{B} = 0, \qquad \nabla \times \mathbf{E} + \frac{\partial B}{\partial t} = 0. \tag{29}$$

We shall adhere to the premise or dictum pointed out by Dyson that the other two Maxwell equations

$$\nabla \cdot \mathbf{E} = 4\pi\rho, \qquad \frac{\partial \mathbf{E}}{\partial t} - \nabla \times \mathbf{H} = 4\pi \mathbf{J}, \tag{30}$$

can never be derived from other basic principles for the reason that they merely define the external charge and current densities. The object of Feynman's approach is to deal with only the electric and magnetic fields, not the sources.

This concludes the demonstration. We accordingly observe that the derivation *a la* Feynman is straightforward as a consequence of the quantum commutation relations as in (11, 12). In particular, the dependence on the parameter β is absent in this formulation.

We may re-express the second relation in (27) in the form

$$\frac{\partial A_j}{\partial x_k} = \frac{1}{2}\epsilon_{mkj} B_m, \tag{31}$$

which will be of use later. We shall now examine an alternative formulation of Feynman's derivation in the context of quantum gravity.

3 Extended Formulation in the Context of Quantum Gravity

We observe that (11, 12) are simplifying *ansatze* which restrict the formulation, in view of (10) and hence needs to be generalized. Accordingly we proceed as follows.

We shall alternatively begin with the commutation relations

$$[x_j, x_k] = 0, \qquad m[x_j, \dot{x}_k] = i\hbar\delta_{jk}, \qquad [x_j, P_k] = i\hbar\delta_{jk}(1+\beta P^2). \tag{32}$$

In this formulation, the fundamental length parameter appears only in the last commutation relation. If we define the vector potential as before, $P_k = m\dot{x}_k + A_k$, we immediately conclude that

$$[x_j, A_k] = i\hbar\delta_{jk}\beta P^2, \tag{33}$$

which is now non-zero as long as $\beta \neq 0$. In other words, we now have $A_k = A_k(x, \dot{x}, t)$ and this alters everything. Proceeding as before, we determine the Lagrangian to be

$$\mathcal{L} = \frac{1}{2}m\dot{x}_l\dot{x}_l + K(x, \dot{x}, t) + \phi(x, t), \tag{34}$$

where

$$K = \int A_k d\dot{x}_k, \tag{35}$$

is an indefinite integral generally depending on $x, \dot{x}, t$ and reduces to $\dot{x}_k A_k(x, t)$ only in the limit $\beta \to 0$. We next consider the other Lagrange equation, (24) and obtain

$$\dot{P}_k = \frac{\partial \mathcal{L}}{\partial x_k} = \frac{\partial \phi}{\partial x_k} + \frac{\partial K}{\partial x_k}. \tag{36}$$

Identifying the force as $F_k = m\ddot{x}_k$, we accordingly obtain the result

$$F_k = \frac{\partial \phi}{\partial x_k} + \frac{\partial K}{\partial x_k} - \frac{\partial A_k}{\partial t} - \frac{\partial A_k}{\partial x_l}\dot{x}_l - \frac{\partial A_k}{\partial \dot{x}_l}\ddot{x}_l. \tag{37}$$

We may now calculate $\partial K/\partial x_k$ from the definition (35) and obtain

$$\frac{\partial K}{\partial x_k} = \frac{1}{2}\epsilon_{mkl}\int B_m d\dot{x}_l. \tag{38}$$

From (31) we also obtain the result

$$\frac{\partial A_k}{\partial x_l}\dot{x}_l = -\frac{1}{2}(\mathbf{v}\times\mathbf{B})_k. \tag{39}$$

Next we need to evaluate $(\partial A_k/\partial \dot{x}_l)\ddot{x}_l$. For this purpose, we need to assume the general functional form of the vector potential. We may assume

$$A_k(x,\dot{x},t) = \epsilon_k C(x) D(\dot{x}) G(t), \tag{40}$$

without loss of generality, where C, D, G are arbitrary functions of the corresponding arguments. This is justified by the fact that $x, \dot{x}, t$ are independent variables. Assuming a Taylor expansion of $D(\dot{x})$, and upon using the relation

$$[x_j, D(\dot{x})] = \sum_{r=0}^{\infty} i\hbar m^{-1}\delta_{jl} r (\dot{x}_l)^{r-1} \left| \left(\frac{\partial}{\partial \dot{x}_l}\right)^r D(\dot{x}) \right|_0, \tag{41}$$

we may calculate, taking into account of the relation in (33), and obtain the result

$$\partial A_k/\partial \dot{x}_l = m\delta_{kl}\beta P^2. \tag{42}$$

Gathering the various results, we thus obtain the equation for the force

$$F_k = \left(\frac{\partial \phi}{\partial x_k} - \frac{\partial A_k}{\partial t}\right) + \frac{1}{2}(\mathbf{v}\times\mathbf{B})_k - \beta P^2 F_k + \frac{1}{2}\epsilon_{mkl}\int B_m d\dot{x}_l. \tag{43}$$

We now see that the magnetic field in general depends on $x, \dot{x}, t$ just as A_k does, while the potential ϕ depends only on x, t. We may rewrite (43) in the form

$$(1+\beta P^2)F_k = E_k + \frac{1}{2}(\mathbf{v}\times\mathbf{B})_k + \frac{1}{2}\epsilon_{mkl}\int B_m d\dot{x}_l. \tag{44}$$

This describes the force in the general case when quantum gravity is present, when $\beta \neq 0$. We observe that this reduces to the familiar result of Sect. 1 in the limit when $\beta \to 0$. The Maxwell equations in the general deformed case are thus in the familiar form but the fields depend on the velocity in addition to the coordinate and time.

We now ask what the meaning of the dependence of the fields on velocity is. This is a difficult question to answer, other than to say that it is a consequence of quantum gravity. The coordinate x_k refers to the coordinate of particles which are sources of the fields, $\dot{x}_k$ their velocities, and thus the fields might depend on the coordinates and velocities of the sources. This makes sense since the space in which the fields exist has the curvature of space-time or the metric determined in general by $x, \dot{x}, t$. One might pursue an explanation of the velocity dependence by a further extension of the recent investigations dealing with the non-commutative geometry framework [13, 14]. However, a formulation involving the metric tensor which would depend on the coordinate and velocity of the particle would introduce additional premises in the formulation. We must again stress the importance of Feynman's point of view, as stated earlier, i.e., make as few assumptions as possible in the pursuit of a universal description of the dynamics. Feynman's purpose was not model building but try to formulate the universality of the dynamics of particles with minimum number of assumptions. For instance we cannot assume the general system to consist of harmonic oscillators because in that case the force is related to the coordinate and we will reproduce the results of the undeformed case. In fact we should refrain from an extensive use of the Hamiltonian and the Hamilton's equations. It thus suffices to simply point out the general form of the Hamiltonian.

We know that the Hamiltonian of the system

$$H = P_k \dot{x}_k - L, \tag{45}$$

which in standard electromagnetic theory is described [15] by

$$H = \frac{(m\dot{x}_k)^2}{2m} - \phi. \tag{46}$$

In the case when quantum gravity is present we find from integration by parts,

$$\int A_k d\dot{x}_k = A_k \dot{x}_k - \int \dot{x}_k \frac{\partial A_k}{\partial \dot{x}_l} d\dot{x}_l = A_k \dot{x}_k - m\beta \int \dot{x}_k P^2 d\dot{x}_l, \tag{47}$$

which shows how the Hamiltonian in general depends on the parameter β. Thus in the case of quantum gravity, we find the Hamiltonian to be given by

$$H = \frac{(m\dot{x}_k)^2}{2m} - \phi + m\beta \int \dot{x}_l P^2 d\dot{x}_l, \tag{48}$$

which shows the explicit dependence on β. Further analysis in terms of Hamilton's equations etc. is of questionable value as stated before.

4 Summary

Feynman's proof, as revealed by Dyson, is a derivation of Maxwell equations of electromagnetism, assuming only Newton's laws of motion obeyed by a particle of classical mechanics and the quantum Heisenberg relation between position and velocity of a single non-relativistic particle. As stated by Dyson, this "proof" might be considered as trivial or incorrect by many physicists. However, one must see Feynman's demonstration in its proper historical light. Feynman's point of view is different, his motivation was to explore as widely as possible the universe of particle dynamics. His object was to make as few assumptions as possible. His final conclusion was that there was no new physics, the road was a dead end and that is the reason for not publishing his results.

In the present work, we have extended this demonstration in a straightforward manner to include quantum gravity. It is demonstrated that this extension is straightforward. Rather than present a formulation based on the metric tensor, we have instead taken the point of view that quantum gravity, as revealed by recent investigations on the subject, can be described by a generalization of the Heisenberg commutation relation signified by the parameter implying a fundamental length. We conclude that when Feynman's derivation is extended in this manner, one can derive Maxwell equations but the fields depend on the coordinates as well as the velocities of the sources via the parameter β signifying a fundamental length. Feynman's purpose was not to introduce many assumptions with a view to model building but instead to explore as widely as possible the formulation of particle dynamics with as few assumptions as possible. We have adhered to Feynman's cautionary attitude and refrained from making too many assumptions in this formulation.

Acknowledgements I would like to dedicate this paper to Professor Ashok Nath Mitra of Delhi University, India, who has been a source of inspiration in my professional life as a theoretical physicist.

References

1. Dyson, F.J.: Am. J. Phys. **58**, 209 (1990)
2. Chang, L., et al.: Phys. Rev. D **65**, 125028 (2002)
3. Chang, L., et al.: Phys. Rev. D **65**, 125027 (2002)
4. Camacho, A.: Int. J. Mod. Phys. D **12**, 1687 (2003)
5. Kempf, A., et al.: Phys. Rev. D **52**, 1108 (1995)
6. Xiang, L., Shen, Y.: Mod. Phys. Lett. A **19**, 1767 (2004)
7. Adler, R., Santiago, D.: Mod. Phys. Lett. A **14**, 1371 (1999)
8. Goldin, G., Sarkar, S.: Preprint arXiv:hep-th/0510107, January (2006)
9. Swamy, P.N.: Int. J. Mod. Phys. A **21**, 6115 (2006)
10. Smith, H.: Introduction to Quantum Mechanics. World Scientific, Singapore (1991)
11. Ohanian, H., Ruffini, R.: In: Gravitation and Spacetime, p. 292. Norton, New York (1976)
12. Mead, C.: Phys. Rev. B 849 (1964)
13. Carinena, J., Figueroa, H.: J. Phys. A **39**, 3763 (2006)
14. Boulahoual, A., Sedra, M.: J. Math. Phys. **44**, 5888 (2003)
15. Goldstein, H.: Classical Mechanics, 2nd edn. Addison-Wesley, Reading (1981)